ON QUASI-THIN ASSOCIATION SCHEMES

MIKHAIL MUZYCHUK AND ILYA PONOMARENKO

Abstract. An association scheme is called quasi-thin if the valency of each its basic relation is one or two. A quasi-thin scheme is Kleinian if the thin residue of it forms a Klein group with respect to the relation product. It is proved that any Kleinian scheme arises from near-pencil on 3 points, or affine or projective plane of order 2. The main result is that any non-Kleinian quasi-thin scheme a) is the two-orbit scheme of a suitable permutation group, and b) is characterized up to isomorphism by its intersection number array. An infinite family of Kleinian quasi-thin schemes for which neither a) nor b) holds is also constructed.

1. Introduction

Given a permutation group $G \leq \mathrm{Sym}(\Omega)$ one can define a *schurian* coherent configuration (Ω, S) where S is the set of G-orbits with respect to the componentwise action of G on the set $\Omega \times \Omega$ (as for a background of association schemes and coherent configurations see Section 2). However, not all coherent configuration can be obtained in this way. This leads naturally to so called *schurity problem*: find an internal characterization of schurian coherent configurations in a given class. Sometimes a solution of this problem is obtained by proving that any coherent configuration from the class is *separable*, i.e. is characterized up to isomorphism by the intersection number array. The *separability problem* consists in finding an internal characterization of separable coherent configurations in a given class. A comprehensive survey of the schurity and separability problems can be found in [7].

In this paper we deal with the schurity and separability problems in the class of quasi-thin association schemes: an association scheme is called *quasi-thin* if the valency of each its basic relation is one or two.[1] Every finite group G of even order with a chosen involution a gives rise to schurian quasi-thin scheme corresponding the action of G on cosets modulo $\langle a \rangle$. Despite of the fact that quasi-thin schemes were introduced explicitly only in 2002 ([13]), the first result about quasi-thin scheme goes back to [17] where it was proved that any primitive quasi-thin scheme is schurian (and, in fact separable). Only quarter of century later this result was generalized to some special classes of quasi-thin schemes [11, 12, 16]. However, nothing was known on their separability. On the other hand, there are non-schurian and non-separable quasi-thin schemes: in the Hanaki-Miyamoto list [9] one can find 1, 1 and 26 non-schurian quasi-thin schemes on 16, 28 and 32 points respectively, and the schemes on 16 and 28 points are non-separable. In all these examples the scheme in question has a very "special structure" explained below.

[1] In [5] it was proved that each *thin* or *regular* association scheme, i.e. such that the valences are ones, is schurian and separable.

Let $\mathcal{X} = (\Omega, S)$ be an association scheme and S_1 the set of all valency one relations in S. We say that $\mathcal{X}$ is a *Kleinian* scheme if the thin residue of it is contained in S_1 and forms a Klein group with respect to the relation product. In this case the degree $n = |\Omega|$ of the scheme $\mathcal{X}$ is divided by 4. Moreover, one can prove that $\mathcal{X}$ is the algebraic fusion of a coherent configuration with $n/4$ regular homogeneous components of degree 4 by means of a group of algebraic automorphisms acting regularly on the set of fibers (see Subsection 9.1). The number n/n_1 where $n_1 = |S_1|$ will be called the *index* of the scheme $\mathcal{X}$. The following statement is the main result of the paper.

Theorem 1.1. *Any non-schurian or non-separable quasi-thin scheme is a Kleinian scheme of index* 4 *or* 7*. Moreover, given* $i \in \{4, 7\}$ *there exist infinitely many both non-schurian and non-separable Kleinian schemes of index* i*.*

The proof is given in Subsection 9.1 and is divided into two parts. In the first of them we prove that all non-Kleinian quasi-thin schemes are schurian and separable. For this purpose we introduce the notion of orthogonal in a quasi-thin scheme and show that any quasi-thin scheme with at most one orthogonal is schurian and separable (Section 5). To deal with the remaining non-Kleinian schemes we study the one point extension of a quasi-thin scheme (Section 6) and give a sufficient condition for such a scheme to be schurian and separable in terms of the existence of the one point extension of an algebraic isomorphism (Theorem 6.5). The key point in the first part of the proof is Theorem 8.1 showing that in our case such extension does always exist.

The second part of the proof deals with a Kleinian quasi-thin scheme. Every such a scheme is an algebraic fusion of a coherent configuration each homogeneous component of which is the scheme of a Klein group. These configurations are called Kleinian and studied in Section 4. We show that any such a configuration is closely related to a partial linear space, and classify all possible spaces in Corollary 4.4. For Kleinian configuration arising from quasi-thin schemes this classification reduced to three cases: near-pencil on three points and affine or projective plane of order 2 (Corollary 9.2). The Kleinian schemes of the near-pencil type are schurian and separable whereas in the other two cases we construct infinitely many non-schurian and non-separable quasi-thin schemes.

Corollary 1.2. *Any non-Kleinian quasi-thin scheme is schurian and separable.* ■

It would be too naive to expect that any commutative quasi-thin scheme is always schurian and separable because such a scheme can be Kleinian. Indeed, let A_1 be the direct product of two cyclic groups of order 4 and f_1 the involutive automorphism of A_1 taking a to a^{-1}; let A_2 be the direct product of two Klein groups and f_2 the involutive automorphism of A_2 which interchanges the coordinates. Denote by $\mathcal{X}_i$ the scheme of the permutation group on A_i generated by the regular representation of A_i and the automorphism f_i, $i = 1, 2$. Then $\mathcal{X}_1$ and $\mathcal{X}_2$ are commutative schurian quasi-thin schemes of degree 16 and rank 10. Moreover, a direct computation shows that they are (a) Kleinian, (b) non-isomorphic and (c) algebraically isomorphic. In particular, none of them is separable. In contrast to this example we prove the following theorem.

Theorem 1.3. *A commutative quasi-thin scheme is schurian.*

The proof of this theorem is reduced by Theorem 1.1 to the case of commutative Kleinian quasi-thin scheme. The schurity of such a scheme is proved by a direct computation in Subsection 9.2.

All undefined terms and notation concerning permutation groups can be found in [2]. To make the paper self-contained we give a background on theory of coherent configurations and on schurity and separability problems in Sections 2 and 3.

Notation. Throughout the paper Ω denotes a finite set. The diagonal of the Cartesian square $\Omega\times\Omega$ is denoted by 1_Ω; for any $\alpha\in\Omega$ we set $1_\alpha=1_{\{\alpha\}}$. For a relation $r\subset\Omega\times\Omega$ we set $r^*=\{(\beta,\alpha):\ (\alpha,\beta)\in r\}$ and $\alpha r=\{\beta\in\Omega:\ (\alpha,\beta)\in r\}$ for all $\alpha\in\Omega$. For $\Gamma,\Delta\subset\Omega$ we set $u_{\Gamma,\Delta}=u\cap(\Gamma\times\Delta)$ and $u_\Gamma=u_{\Gamma,\Gamma}$. For $s\subset\Omega\times\Omega$ we set $r\cdot s=\{(\alpha,\gamma):\ (\alpha,\beta)\in r,\ (\beta,\gamma)\in s \text{ for some } \beta\in\Omega\}$. If S and T are sets of relations, we set $S\cdot T=\{s\cdot t:\ s\in S,\, t\in T\}$. The set of all unions of the elements of S is denote by $S^\cup$.

2. Association schemes and coherent configurations

This section accumulates the basic definitions and facts about coherent configurations and association schemes which are needed for understanding the paper (see also [7, 18]).

2.1. **Definitions.** A pair $\mathcal{X}=(\Omega,S)$ where Ω is a finite set and S a partition of $\Omega\times\Omega$, is called a *coherent configuration* on Ω if $1_\Omega\in S^\cup$, $S^*=S$ and given $u,v,w\in S$, the number

$$c_{uv}^w=|\alpha u\cap\gamma v^*|$$

does not depend on the choice of $(\alpha,\gamma)\in w$. The elements of Ω, S, $S^\cup$ and the numbers (S3) are called the *points*, the *basic relations*, the *relations* and the *intersection numbers* of $\mathcal{X}$, respectively. The numbers $|\Omega|$ and $|S|$ are called the *degree* and *rank* of it. The coherent configuration $\mathcal{X}$ is *commutative* if $c_{uv}^w=c_{vu}^w$ for all $u,v,w\in S$. The unique basic relation containing a pair $(\alpha,\beta)\in\Omega\times\Omega$ is denoted by $r(\alpha,\beta)$.

For the intersection numbers we have the following well-known identities (see [10]).

$$c_{u^*v^*}^{w^*}=c_{vu}^w \quad\text{and}\quad |w|c_{uv}^{w^*}=|u|c_{vw}^{u^*}=|v|c_{wu}^{v^*},\qquad u,v,w\in S$$

If the configuration is homogeneous (a scheme), then these equalities my be rewritten as follows (see [18]):

$$c_{u^*v^*}^{w^*}=c_{vu}^w \quad\text{and}\quad n_wc_{uv}^{w^*}=n_uc_{vw}^{u^*}=n_vc_{wu}^{v^*},\qquad u,v,w\in S \tag{1}$$

The set of basic relations contained in $u\cdot v$ with $u,v\in S^\cup$ is denoted by uv. Sometimes it is useful to treat uv as a multiset in which an element $w\in S$ appears with the multiplicity c_{uv}^w. This multiset will be written as the element of the free module $\mathbb{Z}S$ equipped by the involution $*$ and the natural scalar product defined by

$$(\sum_{s\in S}x_ss)^*=\sum_{s\in S}x_ss^* \quad\text{and}\quad \langle\sum_{s\in S}x_ss,\sum_{s\in S}y_ss\rangle=\frac{1}{|\Omega|}\sum_{s\in S}x_sy_s|s|.$$

Notice that this a scalar product is associative, that is

$$\langle xy,z\rangle=\langle y,x^*z\rangle,\qquad x,y,z\in\mathbb{Z}S. \tag{2}$$

For a homogeneous configuration the scalar product reads as follows:

$$\langle\sum_{s\in S} x_s s, \sum_{s\in S} y_s s\rangle = \sum_{s\in S} x_s y_s n_s.$$

Each time we use notation uv it will be clear is it a set or multiset.

2.2. **Fibers and homogeneity.** Any set $\Delta\subset\Omega$ for which $1_\Delta\in S$, is called the *fiber* of the coherent configuration $\mathcal{X}$; the set of all of them is denoted by $\mathrm{Fib}(\mathcal{X})$. Clearly, the set of points is the disjoint union of fibers. One can also see that if Δ is a union of fibers and S_Δ is the set of all nonempty relations u_Δ with $u\in S$, then (Δ,S_Δ) is a coherent configuration, called the *restriction* of $\mathcal{X}$ to Δ. Besides, for any basic relation $u\in S$ there exist uniquely determined fibers Δ,Γ such that $u\subset\Delta\times\Gamma$. Set

$$n_u = c_{uu^*}^v$$

where $v=1_\Delta$. Then the number $|\delta u| = n_u$ does not depend on $\delta\in\Delta$. When $n_u=n_{u^*}=1$, the relation u is called *thin*.

The coherent configuration $\mathcal{X}$ is called *homogeneous* or a *scheme* if $1_\Omega\in S$, or equivalently if $\mathrm{Fib}(\mathcal{X})=\{\Omega\}$. In this case $n_u=n_{u^*}$ for all $u\in S$; the number n_u is called the *valency* of u. The set of all basic relations of valency m is denoted by S_m. The following result proved in [15] will be used in Section 5.

Lemma 2.1. *Let (Ω,S) be a scheme and $u,v\in S$. Then $c_{u^*v}^w\le 1$ for all $w\in S$ if and only if $uu^*\cap vv^*=\{1_\Omega\}$.* ∎

2.3. **Closed sets.** Let $\mathcal{X}=(\Omega,S)$ be a scheme. A set $T\subset S$ is called *closed*, notation $T\le S$, if $TT^*\subset T$. It is easily seen that the set S_1, called the *thin radical*of $\mathcal{X}$ in [18], is closed (and forms a group with respect to the relational product). The intersection of all closed sets containing the set $\bigcup_{u\in S} uu^*$ is called the *thin residue* of $\mathcal{X}$. The union of all relations from a closed set T is an equivalence relation on Ω with classes αT, $\alpha\in\Omega$. The set of all these classes is denoted by Ω/T. One can prove that the pairs

$$\mathcal{X}_\Delta=(\Delta,S_\Delta)\quad\text{and}\quad \mathcal{X}_{\Omega/T}=(\Omega/T,S_{\Omega/T})$$

are schemes where $\Delta\in\Omega/T$ and S_Δ is as above, and $S_{\Omega/T}$ consists of all relations of the form $\{(\Delta,\Gamma)\in\Omega/T\times\Omega/T:\ u_{\Delta,\Gamma}\ne\emptyset\}$ with $u\in S$. The schemes $\mathcal{X}_\Delta$ and $\mathcal{X}_{\Omega/T}$ are called the *restriction* of $\mathcal{X}$ to Δ, and the *quotient* of $\mathcal{X}$ modulo T.

2.4. **Extensions.** There is a natural partial order $\le$ on the set of all coherent configurations on the set Ω. Namely, given two coherent configurations $\mathcal{X}=(\Omega,S)$ and $\mathcal{X}'=(\Omega,S')$ we set

$$\mathcal{X}\le\mathcal{X}'\ \Leftrightarrow\ S^\cup\subset(S')^\cup.$$

In this case $\mathcal{X}'$ is called an *extension* or *fission* of $\mathcal{X}$. The minimal and maximal elements with respect to that order are respectively the coherent configurations of rank 2 and of rank n^2 where $n=|\Omega|$. The first of them is called *trivial*; its basic relations are 1_Ω and $\Omega\times\Omega\setminus\{1_\Omega\}$. The second one is called *complete*; in this case $S^\cup$ consists of all binary relations on Ω.

Let $\mathcal{X}=(\Omega,S)$ be a coherent configuration and $\alpha\in\Omega$. Denote by S_α the set of basic relations of the smallest coherent configuration on Ω such that

$$1_\alpha\in S_\alpha\quad\text{and}\quad S\subset S_\alpha^\cup.$$

The coherent configuration $\mathcal{X}_\alpha = (\Omega, S_\alpha)$ is called the *α-extension* (or a *one point extension*) of the coherent configuration $\mathcal{X}$. It is easily seen that given $u, v, w \in S$ the set αu and the relation $w_{\alpha u, \alpha v}$ are unions of some fibers and some basic relations of the coherent configuration $\mathcal{X}_\alpha$, respectivelt.

Let $\mathcal{X} = (\Omega, S)$ be a scheme and $T \subset S$ a closed set containing the thin residue of $\mathcal{X}$. Denote by $S_{(T)}$ the set of all relations $u_{\Delta,\Gamma}$ where $u \in S$ and $\Delta, \Gamma \in \Omega/T$. Then from [8, Theorem 2.1] (see also [14]) it follows that the pair $\mathcal{X}_{(T)} = (\Omega, S_{(T)})$ is a coherent configuration; it is called the *thin residue extension* of the scheme $\mathcal{X}$.

2.5. **1-regularity.** Let $\mathcal{X} = (\Omega, S)$ be a coherent configuration. A point $\alpha \in \Omega$ is called *regular* (in $\mathcal{X}$), if

$$|\alpha u| \le 1, \qquad u \in S. \tag{3}$$

Suppose that the set Δ of all regular points is nonempty. Then the coherent configuration $\mathcal{X}$ is called 1*-regular*. In this case all basic relations of the coherent configuration $\mathcal{X}_\Delta$ are thin. A 1-regular scheme is called *regular*; regular schemes are exactly thin schemes in the sense of [18]. One can prove that if $\mathcal{X}$ is a scheme and T is the thin residue of $\mathcal{X}$, then the scheme $\mathcal{X}_{\Omega/T}$ is regular.

2.6. **Direct sum and tensor product.** Let $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ be two coherent configurations. Denote by $\Omega \sqcup \Omega'$ the disjoint union of Ω and Ω', and by $S \boxplus S'$ the union of the set $S \cup S'$ and the set of all relations $\Delta \times \Delta'$ and $\Delta' \times \Delta$ with $\Delta \in \mathrm{Fib}(\mathcal{X})$ and $\Delta' \in \mathrm{Fib}(\mathcal{X}')$. Then the pair

$$\mathcal{X} \boxplus \mathcal{X}' = (\Omega \sqcup \Omega', S \boxplus S')$$

is a coherent configuration called the *direct sum* of $\mathcal{X}$ and $\mathcal{X}'$. One can see that $\mathcal{X} \boxplus \mathcal{X}'$ is the smallest coherent configuration (on $\Omega \sqcup \Omega'$) the restriction of which to Ω and Ω' are respectively $\mathcal{X}$ and $\mathcal{X}'$. It should be noted that the direct sum of any two coherent configurations is non-homogeneous.

Set $S \otimes S' = \{u \otimes u' : \ u \in S, \ u \in S'\}$ where $u \otimes u'$ is the relation on $\Omega \times \Omega'$ consisting of all pairs $((\alpha, \alpha'), (\beta, \beta'))$ with $(\alpha, \beta) \in u$ and $(\alpha', \beta') \in u'$. Then the pair

$$\mathcal{X} \otimes \mathcal{X}' = (\Omega \times \Omega', S \otimes S')$$

is a coherent configuration called the *tensor product* of $\mathcal{X}$ and $\mathcal{X}'$. It should be noted that it is homogeneous if only if so are the factors.

3. Schurian and separable coherent configurations

3.1. **Isomorphisms and schurity.** Two coherent configurations are called *isomorphic* if there exists a bijection between their point sets preserving the basic relations. Any such bijection is called the *isomorphism* of these coherent configurations. The group of all isomorphisms of a coherent configuration $\mathcal{X} = (\Omega, S)$ contains a normal subgroup

$$\mathrm{Aut}(\mathcal{X}) = \{f \in \mathrm{Sym}(\Omega) : \ u^f = u, \ u \in S\}$$

called the *automorphism group* of $\mathcal{X}$. It is easily seen that given $\alpha \in \Omega$ we have $\mathrm{Aut}(\mathcal{X})_\alpha = \mathrm{Aut}(\mathcal{X}_\alpha)$ where $\mathcal{X}_\alpha = (\Omega, S_\alpha)$.

Conversely, let $G \le \mathrm{Sym}(\Omega)$ be a permutation group and S the set of orbits of the componentwise action of G on $\Omega \times \Omega$. Then $\mathcal{X}$ is a coherent configuration and we call it the *coherent configuration of* G. This coherent configuration is homogeneous

if and only if the group is transitive; in this case we say that $\mathcal{X}$ is the *scheme of* G. A coherent configuration on Ω is called *schurian* if it is the coherent configuration of some permutation group on Ω. It is easily seen that a coherent configuration $\mathcal{X}$ is schurian if and only if it is the coherent configuration of the group $\mathrm{Aut}(\mathcal{X})$.

3.2. **Algebraic isomorphisms and separability.** Two coherent configurations $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ are called *algebraically isomorphic* if

$$c_{uv}^{w} = c_{u'v'}^{w'}, \qquad u, v, w \in S, \tag{4}$$

for some bijection $\varphi : S \to S',\ u \mapsto u'$ called the *algebraic isomorphism* from $\mathcal{X}$ to $\mathcal{X}'$. Each isomorphism f from $\mathcal{X}$ to $\mathcal{X}'$ induces in a natural way an algebraic isomorphism between these schemes denoted by φ_f. The set of all isomorphisms inducing the algebraic isomorphism φ is denoted by $\mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$. In particular,

$$\mathrm{Iso}(\mathcal{X}, \mathcal{X}, \mathrm{id}_S) = \mathrm{Aut}(\mathcal{X})$$

where id_S is the identical mapping on S. A coherent configurations $\mathcal{X}$ is called *separable* if for any algebraic isomorphism $\varphi : \mathcal{X} \to \mathcal{X}'$ the set $\mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ is a non-empty one. Given points $\alpha \in \Omega$ and $\alpha' \in \Omega'$ an algebraic isomorphism $\varphi' : \mathcal{X}_\alpha \to \mathcal{X}'_{\alpha'}$ is called the (α, α')-*extension* or the *one point extension* of φ if

$$\varphi'(1_\alpha) = 1_{\alpha'}, \qquad \varphi'(u) \subset \varphi(\widetilde{u}),\ u \in S_\alpha, \tag{5}$$

where $\widetilde{u}$ is the unique basic relation of $\mathcal{X}$ that contains u. Clearly, φ' is uniquely determined by φ.

3.3. **Examples.** One can see that a coherent configuration is 1-regular if and only if it is a coherent configuration of a permutation group having a faithful regular orbit. The proof of this statement as well as the next one can be found in [5].

Theorem 3.1. *Any* 1*-regular coherent configuration is schurian and separable.* ∎

One can define the class of all coherent configurations that can be constructed from 1-regular coherent configurations by means of direct sums and tensor products. By Theorem 3.1 and the following statement proved in [3, Theorems 1.17,1.20] any coherent configuration from this class is schurian and separable.

Theorem 3.2. *Let $\mathcal{X}_1$, $\mathcal{X}_2$ be coherent configurations and let $\mathcal{X}$ be $\mathcal{X}_1 \boxplus \mathcal{X}_2$ or $\mathcal{X}_1 \otimes \mathcal{X}_2$. Then $\mathcal{X}$ is schurian (resp. separable) if and only if both $\mathcal{X}_1$ and $\mathcal{X}_2$ are schurian (resp. separable).* ∎

3.4. **Thin residue extension.** In this section we study the schurity and separability of thin residue extension of an arbitrary scheme.

Theorem 3.3. *Any scheme with the separable thin residue extension is separable, and is schurian if and only if so is the extension.*

Proof. Let $\mathcal{X} = (\Omega, S)$ be a scheme and T its thin residue. Then from [8, Theorem 2.1] it follows that the following statements hold:

(i) given $f \in \mathrm{Aut}(\mathcal{X}_{\Omega/T})$ the mapping $\psi_f : u_{\Delta,\Gamma} \mapsto u_{\Delta^f,\Gamma^f}$ is an algebraic isomorphism of the coherent configuration $\mathcal{X}_{(T)}$,[2]

[2] This statement was also proved in [14].

(ii) given an algebraic isomorphism $\varphi : \mathcal{X} \to \mathcal{X}'$ and $f \in \mathrm{Iso}(\mathcal{X}_{\Omega/T}, \mathcal{X}'_{\Omega'/T'}, \varphi_{\Omega/T})$ where $T' = T^\varphi$, there exists an algebraic isomorphism $\varphi_T : \mathcal{X}_{(T)} \to \mathcal{X}'_{(T')}$ extending φ and such that $(\varphi_T)_{\Omega/T}$ is induced by f.

Suppose that the thin residue extension $\mathcal{X}_0 = \mathcal{X}_{(T)}$ of the scheme $\mathcal{X}$ is separable. Then the first statement immediately follows from statement (ii). To prove the second one suppose first that the coherent configuration $\mathcal{X}_0$ is schurian. Take a relation $u \in S$ and pairs (α_1, β_1), $(\alpha_2, \beta_2) \in u$. Set

$$\Delta_i = \alpha_i T \quad \text{and} \quad \Gamma_i = \beta_i T, \qquad i = 1, 2.$$

Then the pairs (Δ_1, Γ_1) and (Δ_2, Γ_2) belong to the same relation of the quotient scheme $\mathcal{X}_{\Omega/T}$. Since this scheme is regular (and hence schurian), one can find an automorphism $f \in \mathrm{Aut}(\mathcal{X}_{\Omega/T})$ taking (Δ_1, Γ_1) to (Δ_2, Γ_2). By statement (i) it induces the algebraic isomorphism ψ_f of the coherent configuration $\mathcal{X}_0$ such that

$$(\Delta_1)^{\psi_f} = \Delta_2 \quad \text{and} \quad (\Gamma_1)^{\psi_f} = \Gamma_2. \tag{6}$$

Since this coherent configuration is separable, the algebraic isomorphism ψ_f is induced by an isomorphism g of $\mathcal{X}_0$ to itself. From the definition of ψ_f it follows that $g \in \mathrm{Aut}(\mathcal{X})$. Moreover, due to (6) we also have

$$(\Delta_1)^g = \Delta_2 \quad \text{and} \quad (\Gamma_1)^g = \Gamma_2.$$

Thus without loss of generality we can assume that $\Delta_1 = \Delta_2$ and $\Gamma_1 = \Gamma_2$. Denote these sets by Δ and Γ. Then the pairs (α_1, β_1) and (α_2, β_2) belong to the relation $u_{\Delta,\Gamma}$ and we are done by the schurity of $\mathcal{X}_0$.

To complete the proof suppose that the scheme $\mathcal{X}$ is schurian. Take a basic relation u_0 of the coherent configuration $\mathcal{X}_0$ and pairs (α, β), $(\alpha', \beta') \in u_0$. Then u_0 is contained in a certain relation $u \in S$, and

$$(\alpha, \beta),\ (\alpha', \beta') \in u \quad \text{and} \quad \alpha T = \alpha' T,\ \beta T = \beta' T.$$

Denote the latter two sets (which are elements of Ω/T) by Δ and Γ. By the schurity of $\mathcal{X}$ one can find $f \in \mathrm{Aut}(\mathcal{X})$ such that $(\alpha^f, \beta^f) = (\alpha', \beta')$. Clearly, $\Delta^f = \Delta$ and $\Gamma^f = \Gamma$. On the other hand, since the scheme $\mathcal{X}_{\Omega/T}$ is regular, we have

$$\mathrm{Aut}(\mathcal{X}_0) = \mathrm{Aut}(\mathcal{X})_{\{\Delta\}} \cap \mathrm{Aut}(\mathcal{X})_{\{\Gamma\}}.$$

Thus $f \in \mathrm{Aut}(\mathcal{X}_0)$, and the coherent configuration $\mathcal{X}_0$ is schurian. ∎

The assumption on the separability of the thin residue extension in Theorem 3.3 is essential. Indeed, let $\mathcal{X}$ be the quasi-thin scheme of degree 16 that has number #173 in [9]. Then the group $\mathrm{Aut}(\mathcal{X})$ has two orbits, and hence the scheme $\mathcal{X}$ is non-schurian. On the other hand, its thin residue extension is schurian but non-separable.

4. Klein configurations

4.1. **Definition and structure.** Throughout this section we fix a Klein group G. A coherent configuration $\mathcal{X} = (\Omega, S)$ is a *Klein configuration* if any its homogeneous component is the scheme of a regular permutation group isomorphic to G. In this case we fix a semiregular action of G on Ω such that $\mathrm{Orb}(G, \Omega) = \mathrm{Fib}(\mathcal{X})$ and the homogeneous component $\mathcal{X}_\Delta$ corresponding to a fiber $\Delta \in \mathrm{Fib}(\mathcal{X})$ is the scheme of

the group G^Δ. This semiregular action of G is completely determined by the group isomorphisms

$$G \to S_{\Delta,\Delta},\ g \mapsto g_\Delta \tag{7}$$

where given $\Gamma, \Delta \in \mathrm{Fib}(\mathcal{X})$ we set $S_{\Gamma,\Delta} = \{s \in S :\ s \subset \Gamma \times \Delta\}$. It should be noted that these isomorphisms, and hence the semiregular action of the group G, can be chosen not a unique way.

Let $\mathcal{X} = (\Omega, S)$ be a Klein configuration and $\mathrm{Fib}(\mathcal{X}) = \{\Omega_i\}_{i\in I}$ where I is a nonempty finite set. Then due to [4, Lemma 5.1] given indices $i, j \in I$ the groups

$$L_{ij} = \{g \in G :\ g_i \cdot s = s\}, \quad R_{ij} = \{g \in G :\ s \cdot g_j = s\} \tag{8}$$

do not depend on the relation $s \in S_{ij}$ where $g_i = g_{\Omega_i}$, $g_j = g_{\Omega_j}$ and $S_{ij} = S_{\Omega_i,\Omega_j}$. Moreover, from the same result it follows that $L_{ij} = R_{ji}$, $R_{ij} = L_{ji}$ and exactly one of the following three statements hold:

(K1) $L_{ij} = R_{ij} = G$ and $S_{ij} = \{\Omega_i \times \Omega_j\}$,
(K2) $L_{ij} = R_{ij} = \{1\}$ and $S_{ij} = \mathrm{Orb}(G, \Omega_i \times \Omega_j)$,
(K3) $|L_{ij}| = |R_{ij}| = 2$ and $S_{ij} = \{\Omega_{i,1} \times \Omega_{j,1^f} \cup \Omega_{i,2} \times \Omega_{j,2^f}\}_{f\in\mathrm{Sym}(2)}$

where $\{\Omega_{i,1}, \Omega_{i,2}\} = \mathrm{Orb}(L_{ij}, \Omega_i)$ and $\{\Omega_{j,1}, \Omega_{j,2}\} = \mathrm{Orb}(R_{ij}, \Omega_j)$. In what follows the array $R = R(\mathcal{X}, G) = (R_{ij})$ is treated as a matrix whose rows and columns are indexed by the elements of the set I. Clearly, given $i, j, k \in I$ we have

$$R_{ji} = R_{ki} \quad \& \quad |R_{jk}| = |R_{ji}| = 2 \qquad \Rightarrow \qquad R_{ij} = R_{kj} \quad \& \quad R_{ik} = R_{jk}. \tag{9}$$

(Indeed, we always have $S_{ji} \cdot S_{ik} \subset S_{jk}^{\cup}$, and the left-hand side conditions imply that in fact $S_{ji} \cdot S_{ik} = S_{jk}$.) Moreover, the relation $\sim$ consisting of all pairs $(i, j) \in I \times I$ such that $R_{ij} = \{1\}$, is an equivalence relation, and

$$j \sim k \ \Rightarrow\ R_{ji} = R_{ki}, \qquad i \in I. \tag{10}$$

It should be noted that the matrix R depends on the choice of isomorphisms (7). On the other hand, the conditions (K1), (K2) and (K3) imply the following statement.

Lemma 4.1. *Let $\mathcal{X}$ and $\mathcal{X}'$ be Klein configurations on the same set. Suppose that $\mathrm{Fib}(\mathcal{X}) = \mathrm{Fib}(\mathcal{X}')$ and $R(\mathcal{X}, G) = R(\mathcal{X}', G)$. Then $\mathcal{X} = \mathcal{X}'$.* ■

4.2. **Reduced configurations.** Given a set $J \subset I$ denote by $\mathcal{X}_J$ the restriction of the Klein configuration $\mathcal{X}$ to the union Ω_J of all fibers Ω_i, $i \in J$. Then $\mathcal{X}_J$ is also a Klein configuration and $R_J = R(\mathcal{X}_J, G)$ is a submatrix of R the rows and columns of which are the elements of J. Denote by $\mathcal{J}(\mathcal{X}, G)$ the set of all transversals of the equivalence relation $\sim$.

Lemma 4.2. *Given $J \in \mathcal{J}(\mathcal{X}, G)$ the Klein configurations $\mathcal{X}$ and $\mathcal{X}_J$ are schurian (or separable) simultaneously.*

Proof. By the lemma hypotheses for any fiber Ω_i with $i \in I \setminus J$ any relation of the set S_{ij} with $j \sim i$ is thin. Thus the required statement immediately follows from statement (2) of [6, Lemma 9.4]. ■

The Klein configuration $\mathcal{X}_J$ from Lemma 4.2 is *reduced*: by the definition this means that the equivalence relation $\sim$ is trivial, or equivalently $|R_{ij}| \geq 2$ for all distinct $i, j \in I$. For any reduced configuration we define the incidence structure $\mathcal{G} = (I, L)$ with the point set I and the line set L consisting of all sets

$$L_i(H) = \{i\} \cup \{j \in I :\ R_{ji} = H\}$$

where $H = R_{ji}$ for some $j \in I \setminus \{i\}$. (Here H is always a subgroup of G of order 2.) In the following statement we show that under rather a technical assumption the geometry $\mathcal{G}$ is a partial linear space, i.e. an incidence structure such that the lines have size at least 2 and two distinct points are incident to at most one line.

Lemma 4.3. *Let $\mathcal{X}$ be a reduced Klein configuration such that for any $i \in I$ there exists $j \in I \setminus \{i\}$ with $R_{ji} \neq G$. Then $\mathcal{G}$ is a partial linear space in which any point is incident to at most three lines.*

Proof. By the hypothesis any line $L_i(H) \in L$ contains at least two points: i and $j \in I \setminus \{i\}$ for which $R_{ji} \neq G$. Next, from (9) it follows that given an element $i \in I$ and a group $H \leq G$ of order 2 we have

$$j \in L_i(H) \;\Rightarrow\; L_j(K) = L_i(H) \tag{11}$$

where $K = R_{ij}$. Next, suppose that distinct points i and j are incident to two lines $L_k(H)$ and $L_{k'}(H')$ where $k, k' \in I$ and $H, H' \leq G$ are of order 2. Then due to (9) we have $R_{ki} = R_{ji} = R_{k'i}$. Denote this group by K. Clearly, $|K| = 2$. Therefore due to (11) we conclude that $L_k(H) = L_i(K) = L_{k'}(H')$. Thus any two distinct points are incident to at most one line. Since the group G has exactly three subgroups of order 2, any point is incident to at most three lines. ∎

We recall that a linear space is partial linear space in which any two distinct points are incident to exactly one line.

Corollary 4.4. *In the condition of Lemma 4.3 suppose that $R_{ij} \neq G$ for all $i, j \in I$. Then either $|L| = 1$, or $|I| \leq 7$ and $\mathcal{G}$ is a projective or affine plane of order 2, or $\mathcal{G}$ is one of the four linear spaces at Fig.1.*[3]

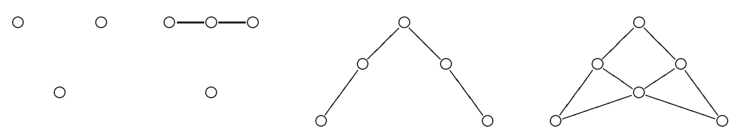

FIGURE 1. The linear spaces with ≤ 7 points, and ≥ 2 lines of size ≤ 3, in which the union of all lines incident to a point coincides with the point set

Proof. To prove the second statement suppose that $R_{ij} \neq G$ for all $i, j \in I$. Then (a) two distinct points of $\mathcal{G}$ are incident to exactly one line, i.e. $\mathcal{G}$ is a linear space, and (b) the union of all lines incident to a point coincides with I. Without loss of generality we can assume that $|L| \geq 2$. This implies that each line is incident at most 3 points (for otherwise, any point not in the line is incident to at least 4 points in contrast to the first statement). Thus $|I| \leq 7$ and the required statement follows from the list of linear spaces on at most 9 points given in [1, pp.190-191]. ∎

The first linear space at Fig. 1 is known as *near-pencil* on 3 points.

[3]In the diagrams we omit all 2-point lines.

5. Quasi-thin schemes. Orthogonals

A scheme $\mathcal{X} = (\Omega, S)$ is called *quasi-thin* if $S = S_1 \cup S_2$. In such a scheme the product of two basic relations is again a basic relation unless both of them are *thick*, i.e. belong to S_2. By [11, Lemma 4.1] given a thick relation u there exists the uniquely determined basic relation $u^\perp$ such that

$$uu^* = \{1_\Omega, u^\perp\}.$$

This relation is called the *orthogonal* of u. It is easily seen that any orthogonal is a (non-reflexive) symmetric relation. The following statement was proved in [12].

Lemma 5.1. *Given thick relations u and v in the set S we have*

1. *$u^\perp = v^\perp$ and $u^\perp \in S_1$ if and only if either $u^*v = 2a + 2b$ with $a, b \in S_1$, or $u^*v = 2a$ with $a \in S_2$,*
2. *$u^\perp = v^\perp$ and $u^\perp \notin S_1$ if and only if $u^*v = 2a + b$ with $a \in S_1$ and $b \in S_2$,*
3. *$u^\perp \neq v^\perp$ if and only if $u^*v = a + b$ with $a, b \in S_2$.* ■

Given $T \subset S_2$ we set $T^\perp = \{u^\perp : u \in T\}$. Any element from the set $S^\perp$ is called an orthogonal of the scheme $\mathcal{X}$.

Theorem 5.2. *Any quasi-thin scheme with at most one orthogonal is schurian and separable.*

Proof. Let $\mathcal{X} = (\Omega, S)$ be a quasi-thin scheme. If $S^\perp = \emptyset$, then this scheme is regular, and hence 1-regular. Therefore it is schurian and separable by Theorem 3.1. Thus we can assume that $S^\perp = \{u\}$ for some non-reflexive basic relation u. Then $u = u^*$ and $v^*v \subset \{1_\Omega, u\}$ for all $v \in S$. Therefore the set $\{1_\Omega, u\}$ is closed and coincides with the thin residue T of the scheme $\mathcal{X}$.

Suppose first that $u \in S_2$. Then $u^\perp = u$ is a thick relation. By statement (2) of Lemma 5.1 this implies that $S_2 = S_1 u = uS_1$. Therefore $S = S_1 T$. Since $S_1 \cap T = \{1_\Omega\}$ and $|T| = 2$, it follows that

$$\mathcal{X} \cong \mathcal{X}_{\alpha S_1} \otimes \mathcal{X}_{\alpha T}, \qquad \alpha \in \Omega.$$

However, the scheme $\mathcal{X}_{\alpha S_1}$ is regular whereas the scheme $\mathcal{X}_{\alpha T}$ is trivial. Thus both of these scheme are schurian and separable, and we are done by Theorem 3.2.

Let $u \in S_1$. Then $|\alpha T| = 2$ for all $\alpha \in \Omega$. Since every fiber of the thin residue extension $\mathcal{X}_0 = \mathcal{X}_{(T)}$ is of the form αT, it follows that given $\Delta, \Gamma \in \mathrm{Fib}(\mathcal{X}_0)$ the set $\Delta \times \Gamma$ is either a basic relation of $\mathcal{X}_0$, or the union of two thin basic relations of $\mathcal{X}_0$. In the latter case we will write $\Delta \sim \Gamma$. It is easily seen that $\sim$ is an equivalence relation on the set $\mathrm{Fib}(\mathcal{X}_0)$. Denote by I the set of its classes, and given $i \in I$ set Ω_i to be the union of fibers belonging the class i. Then

$$\mathcal{X}_0 = \boxplus_{i \in I} \mathcal{X}_i$$

where $\mathcal{X}_i = (\mathcal{X}_0)_{\Omega_i}$. Any summand here is a 1-regular coherent configuration, and hence is schurian and separable. By Theorem 3.2 this implies that so is the coherent configuration $\mathcal{X}_0$. Thus the scheme $\mathcal{X}$ is schurian and separable by Theorem 3.3. ■

From [17, pp.71,72] it follows that any primitive[4] quasi-thin scheme is schurian and separable. Moreover, an inspection of the Hanaki-Miyamoto list [9] shows that

[4] A scheme on Ω is called primitive if any equivalence relation on Ω that is a union of basic relations, is 1_Ω or $\Omega \times \Omega$.

any imprimitive quasi-thin scheme of degree at most 8 has at most one orthogonal. Thus by Theorem 5.2 we have the following statement.

Corollary 5.3. *Any quasi-thin scheme of degree at most* 8 *is schurian and separable.* ∎

We recall that a scheme is called *Kleinian* if its thin residue consists of thin relations and forms a Klein group with respect to the relation product. In the following statement these schemes are characterized by means of orthogonals. Below given $u \in S$ we set

$$S_u = \{v \in S_2 : \; v^\perp = u\}.$$

Clearly $S_u S_1 = S_u$.

Lemma 5.4. *A quasi-thin scheme $\mathcal{X}$ is Kleinian if and only if $S^\perp \subset S_1$ and either $|S^\perp| = 2$ or $|S^\perp| = 3$ and the set $\{1_\Omega\} \cup S^\perp$ is closed. If $\mathcal{X}$ is a commutative Kleinian scheme, then $|S^\perp| = 3$.*

Proof. The necessity is obvious. To prove the sufficiency without loss of generality we can assume that $\mathcal{X}$ is a quasi-thin scheme with exactly two thin orthogonals u and v. Then given a thick relation $x \in S_u$, the relation $vx \in S$ is also thick and

$$(vx)(vx)^* = v(xx^*)v = v\{1_\Omega, u\}v = \{1_\Omega, vuv\}.$$

Since vuv is also a basic relation, we conclude that $vuv = (vx)^\perp \in \{u, v\}$. Therefore $vuv = u$, i.e. u and v commute. Thus the thin residue of $\mathcal{X}$ coincides with the group $\langle u, v\rangle = \{1_\Omega, u, v, uv\}$ which in our case is obviously the Klein group.

To complete the proof suppose on the contrary that $\mathcal{X}$ is a commutative Kleinian scheme with exactly two orthogonals u and v. Then

$$S = S_1 \cup S_u \cup S_v \quad \text{and} \quad S_u^* = S_u, \; S_v^* = S_v. \tag{12}$$

Moreover, given basic relations x and y such that $x^\perp = y^\perp$, and any $z \in xy$ we have $zz^* \subset (xy)(xy)^* = xx^*yy^* \subset \{1_\Omega, x^\perp\}$. Since also $S_u = S_u S_1 = S_1 S_u$ and $S_v = S_v S_1 = S_1 S_v$, we see that $S_1 \cup S_u$ and $S_1 \cup S_v$ are closed subsets of $\mathcal{X}$ the union of which equals S. Therefore one of them coincides with S. A contradiction. ∎

Let $u \neq v$ be thick basic relations of a quasi-thin scheme $\mathcal{X}$. We say that they are *adjacent*, $u \approx v$, if $|u^*v| = 2$. Since $|u^*v| = |v^*u|$, the adjacency relation is symmetric. Notice that by Lemma 5.1 the cardinality of $|u^*v|$ is either one or two. Therefore two relations $u, v \in S_2$ are non-adjacent if and only if $|u^*v| = 1$. The following special statement will be used in the proof of Theorem 8.1.

Lemma 5.5. *Let (Ω, S) be a quasi-thin non-Kleinian scheme of degree ≥ 9 and with at least two orthogonals. Suppose that a set $T \subsetneq S_2$ is such that*

$$|T^\perp| \leq 2 \qquad \text{and} \qquad |T_u| \leq 2 \quad \text{for all} \quad u \in T^\perp \cap S_2 \tag{13}$$

where $T_u = T \cap S_u$. Then there exists a relation $t \in S_2 \setminus T$ adjacent to each element of T.

Proof. We observe that if $|T^\perp| < |S^\perp|$, then by statement (3) of Lemma 5.1 the required statement holds for any relation $t \in S$ such that $t^\perp \in S^\perp \setminus T^\perp$. Thus without loss of generality we can assume that $|T^\perp| = |S^\perp|$. Since $|T^\perp| \leq 2$ and $|S^\perp| \geq 2$, this implies that $T^\perp = S^\perp$ and $|S^\perp| = 2$. If $S^\perp \subseteq S_2$, then by statement (3) of Lemma 5.1 any two elements of S_2 are adjacent, and we are done with arbitrary

$t \in S_2 \setminus T$. Moreover, taking into account that the scheme (Ω, S) is non-Kleinian, we conclude by Lemma 5.4 that $S^\perp \not\subset S_1$. Thus we can assume that

$$S^\perp = \{u, v\}, \quad u \in S_2, \quad v \in S_1.$$

If $S_u \setminus T_u \neq \emptyset$, then there exists a relation $t \in S_u \setminus T_u$. By statement (3) of Lemma 5.1 this relation is adjacent to every element of $S_2 \setminus \{t\}$, and we are done. Thus we may assume that $S_u \subseteq T_u$, or equivalently, $S_u = T_u$. To complete the proof we have to verify that the equality

$$|S_u| = 2 \tag{14}$$

leads to a contradiction. Indeed, let us fix a relation $t \in T_u$. Then $rt = st$ for some thin relations r and s, only if they are equal (here $s^*r \in tt^* = \{1_\Omega, u\}$ and hence $s^*r = 1_\Omega$ because $n_u = 2$). Moreover, since u is thick, we have $S_1 S_u = S_u$. Therefore from (14) it follows that $|S_1| = |S_1 t| \le |S_1 S_u| = |S_u| = 2$. This implies that

$$S_1 = \{1_\Omega, v\} \quad \text{and} \quad S_u = \{t, vt\}.$$

Suppose that $u^\perp = u$. Then $S_u = \{u, vu\}$ and $S_u^* = S_u$. It follows that the set $Q = S_1 \cup S_u$ is closed. However, the set $R = S_1 \cup S_v$ is also closed. Thus the set S is a union of the closed subsets Q and R. This implies that one of them coincides with S which is impossible. Thus $u^\perp = v$. Then one can check that

$$tv = vt \quad \text{and} \quad t^* \in \{t, vt\}.$$

So $Q = \langle t \rangle = S_1 \cup S_u \cup \{u\}$ is a closed set and $n_Q = 8$. Again the set S is a union of the closed subsets Q and R. This implies that $S = Q$ whence it follows that $|\Omega| = n_S = n_Q = 8$. Contradiction. ∎

6. One-point extension of a quasi-thin scheme

In this section we first compute the fibers of a one-point extension of a quasi-thin scheme, then analyze its basic relations, and finally give a sufficient condition for its schurity and separability.

Theorem 6.1. *Let $\mathcal{X} = (\Omega, S)$ be a quasi-thin scheme and $\alpha \in \Omega$. Then each fiber of the coherent configuration $\mathcal{X}_\alpha$ is of the form αu, $u \in S$. In particular,*

$$S_\alpha = \bigcup_{u,v \in S} S_\alpha(u, v)$$

where $S_\alpha(u, v) = \{a \in S_\alpha : \ a \subset \alpha u \times \alpha v\}$.

Proof. The second statement immediately follows from the first one. To prove the latter let us define an involution $f_\alpha \in \mathrm{Sym}(\Omega)$ so that

$$\beta^{f_\alpha} = \begin{cases} \beta, & \text{if } r(\alpha,\beta) \in S_1, \\ \beta', & \text{if } r(\alpha,\beta) \in S_2, \end{cases} \tag{15}$$

where β' is defined from the condition $\{\beta, \beta'\} = \alpha r(\alpha, \beta)$. It was proved in [13, Lemma 3.5] that $f_\alpha \in \mathrm{Aut}(\mathcal{X})$ for all α. To complete the proof, let Δ be a fiber of the coherent configuration $\mathcal{X}_\alpha$. Then obviously $\Delta \subset \alpha u$ for some $u \in S$. On the other hand, the set αu is the orbit of the group

$$\langle f_\alpha \rangle \le \mathrm{Aut}(\mathcal{X}_\alpha).$$

Thus $\Delta \supset \alpha u$. Since the converse inclusion is trivial, we conclude that $\Delta = \alpha u$ and we are done. ∎

The conclusion of Theorem 6.1 holds for any schurian scheme, and together with the transitivity of the automorphism group implies the schurity of the scheme in question. Thus as a consequence of that theorem we obtain the following well-known statement [12].

Corollary 6.2. *A quasi-thin scheme $\mathcal{X}$ is schurian if and only if the group* $\mathrm{Aut}(\mathcal{X})$ *is transitive.* ∎

Let $\mathcal{X} = (\Omega, S)$ be a quasi-thin scheme, $\alpha \in \Omega$ and $u, v \in S$. From Theorem 6.1 it follows that $1_{\alpha u} \in S_\alpha$. Therefore given $a \in S_\alpha(u,v)$ the number $|\beta a| = c_{aa^*}^{1_{\alpha u}}$ does not depend on $\beta \in \alpha u$. Since $|\alpha u| \le 2$ and $|\alpha v| \le 2$, this implies that

$$S_\alpha(u,v) = \{\alpha u \times \alpha v\} \quad \text{or} \quad S_\alpha(u,v) = \{f_1, f_2\} \tag{16}$$

where f_1 and f_2 are the two distinct bijections from αu onto αv (treated as binary relations on $\alpha u \times \alpha v$). Thus the set $S_\alpha(u,v)$ consists of one or two elements, and the latter holds only if $|\alpha u| = |\alpha v| = 2$. In this case the the element of $S_\alpha(u,v)$ other than $a = f_i$ is denoted by $\overline{a}$.

Let u and v be basic relations of the scheme $\mathcal{X}$. Due to (1) the intersection numbers c_{uw}^v and $c_{u^*v}^w$ are zero or not simultaneously. Therefore given $\alpha \in \Omega$ the cardinality of the set

$$S(u,v;\alpha) = \{w_{\alpha u, \alpha v} : \ w \in S, \ c_{uw}^v \ne 0\} \tag{17}$$

equal the number $|u^*v|$, and hence does not depend on α. The above set consists of non-empty and pairwise disjoint relations from $S_\alpha(u,v)^\cup$, the union of which coincides with the set $\alpha u \times \alpha v$. It is easily seen that $S(u,v;\alpha)^* = S(v,u;\alpha)$.

Lemma 6.3. *Let (Ω, S) be a quasi-thin scheme with at least two orthogonals, $\alpha \in \Omega$ and $u, v \in S$. Then $S_\alpha(u,v) \ne S(u,v;\alpha)$ if and only if $u \not\approx v$. Moreover, in this case $u^\perp = v^\perp \in S_1$ and*

$$S_\alpha(u,v) = S(u,w;\alpha) \cdot S(w,v;\alpha) \tag{18}$$

for any $w \in S_2$ with $w^\perp \ne u^\perp$.

Proof. Both $S_\alpha(u,v)$ and $S(u,v;\alpha)$ forms a partition of the set $\alpha u \times \alpha v$, and the former partition is a refinement of the latter one. Therefore

$$2 \ge |S_\alpha(u,v)| \ge |S(u,v;\alpha)| = |u^*v|. \tag{19}$$

Thus $S_\alpha(u,v) \ne S(u,v;\alpha)$ if and only if $|S_\alpha(u,v)| = 2$ and $|u^*v| = 1$. Due to (16) the first equality holds only if $u, v \in S_2$, whereas the second one means that $u \not\approx v$. This proves the necessity of the first statement.

Conversely, suppose that u and v are non-adjacent elements of S_2. Then $u^\perp = v^\perp$ by Lemma 2.1. To complete the proof, let $w \in S_2$ be such that $w^\perp \ne u^\perp$. Then statement (3) of Lemma 5.1 implies that $|u^*w| = 2$, and hence by (19) with $v = w$, we obtain that $|S_\alpha(u,w)| = 2$. Similarly, $|S_\alpha(v,w)| = 2$. Due to (16) this implies

$$S_\alpha(u,w) = \{f_1, f_2\} \quad \text{and} \quad S_\alpha(w,v) = \{g_1, g_2\}$$

where f_1 and f_2 (resp. g_1 and g_2) are the two bijections from αu to αw (resp. from αw to αv). Since obviously $S(u,w;\alpha) \cdot S(w,v;\alpha) \subset S_\alpha(u,v)^\cup$ and

$$f_i \cdot g_j = f_i g_j, \qquad i,j = 1,2,$$

this implies that $f_i g_j \in S_\alpha(u,v)$. This proves equality (18), and also the sufficiency of the first statement. ∎

Corollary 6.4. *Let $\mathcal{X}$ be a quasi-thin scheme on Ω with at least two orthogonals. Then given a point $\alpha \in \Omega$ the coherent configuration $\mathcal{X}_\alpha$ is 1-regular.*

Proof. Denote by S the set of basic relations of the scheme $\mathcal{X}$. Let us verify that any point $\beta \in \alpha S_2$ is regular (see Subsection 2.5). To do this let $a \in S_\alpha$ be such that $\beta a \ne \emptyset$. Then by Theorem 6.1 there exist relations $u \in S_2$ and $v \in S$ such that

$$\beta \in \alpha u \quad \text{and} \quad a \in S_\alpha(u,v).$$

Without loss of generality we can assume that $v \in S_2$ (otherwise $|\beta a| = |\alpha v| = 1$ and we are done). Then due to (16) it suffices to verify that $|S_\alpha(u,v)| = 2$. However, this is true by Lemma 6.3: if $S_\alpha(u,v) = S(u,v;\alpha)$, then $|S_\alpha(u,v)| = |u^* v| = 2$, whereas if $S_\alpha(u,v) \ne S(u,v;\alpha)$, then $2 \ge |S_\alpha(u,v)| > |u^* v| = 1$. ∎

The conclusion of Corollary 6.4 is not true when $|S^\perp| = 1$. Indeed, denote by $\mathcal{X}$ the scheme of the wreath product of two regular schemes of degrees 2 and $n \ge 3$. Then $\mathcal{X}$ is a quasi-thin scheme of degree $2n$ with exactly one orthogonal. On the other hand, any point extension of $\mathcal{X}$ is the coherent configuration of the elementary abelian group of order 2^{n-1} with two fixed points and $n-1 \ge 2$ orbits of cardinality 2. It follows that the point extension of $\mathcal{X}$ has no regular points, and hence is not 1-regular.

Theorem 6.5. *Let $\mathcal{X}$ be a quasi-thin scheme with at least two orthogonals. Suppose that any algebraic isomorphism φ from $\mathcal{X}$ to another scheme $\mathcal{X}'$ has one point extension $\varphi_{\alpha,\alpha'} : \mathcal{X}_\alpha \to \mathcal{X}'_{\alpha'}$ for any pair of points $\alpha \in \Omega$ and $\alpha' \in \Omega'$. Then the scheme $\mathcal{X}$ is schurian and separable.*

Proof. By Corollary 6.4 the coherent configuration $\mathcal{X}_\alpha$ is 1-regular. Together with Theorem 3.1 this implies that the set $\mathrm{Iso}(\mathcal{X}_\alpha, \mathcal{X}'_{\alpha'}, \varphi_{\alpha,\alpha'})$ is not empty. Since

$$\mathrm{Iso}(\mathcal{X}_\alpha, \mathcal{X}'_{\alpha'}, \varphi_{\alpha,\alpha'}) \subset \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi),$$

the set $\mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ is also not empty. Thus the scheme $\mathcal{X}$ is separable.

To prove schurity of $\mathcal{X}$ take $\alpha, \alpha' \in \Omega$. Then by the theorem hypothesis the trivial algebraic isomorphism $\mathrm{id}_S : \mathcal{X} \to \mathcal{X}$ has the (α,α')-extension, say $\varphi_{\alpha,\alpha'}$. Since the coherent configuration $\mathcal{X}_\alpha$ is 1-regular (Corollary 6.4), from Theorem 3.1 it follows that there exists an isomorphism

$$f_{\alpha,\alpha'} \in \mathrm{Iso}(\mathcal{X}_\alpha, \mathcal{X}_{\alpha'}, \varphi_{\alpha,\alpha'}).$$

By the definition of $\varphi_{\alpha,\alpha'}$ this isomorphism takes α to α' and preserves every basic relation of $\mathcal{X}$. Therefore $f_{\alpha,\alpha'} \in \mathrm{Aut}(\mathcal{X})$. Since α and α' are arbitrary points of Ω, this means that the group $\mathrm{Aut}(\mathcal{X})$ is transitive. Thus schurity of $\mathcal{X}$ follows from statement (2) of Theorem 6.2. ∎

7. Triangles in a quasi-thin scheme

Let $\mathcal{X} = (\Omega, S)$ be a quasi-thin scheme. A 3-subset T of S_2 is called a *triangle* (in $\mathcal{X}$) if any two distinct elements of T are adjacent. From Lemma 6.3 it follows that any 3-set $T \subset S_2$ with $|T^\perp| = 3$ is a triangle. The following statement which is an immediate consequence of (16), shows that any triangle induces a regular coherent configuration with three fibers of size 2 on the neighborhood of each point.

Lemma 7.1. *Let $\{u,v,w\}$ be a triangle in the scheme $\mathcal{X}$ and $\alpha \in \Omega$. Then $x \cdot y = \overline{x} \cdot \overline{y}$ and $\overline{x} \cdot y = x \cdot \overline{y} = \overline{x \cdot y}$ for all $x \in S_\alpha(u,w)$ and $y \in S_\alpha(w,v)$.* ∎

We say that a triangle $\{u,v,w\}$ is *exceptional* if $u^\perp \cdot v^\perp \cdot w^\perp = 1_\Omega$. Notice that in this case $u^\perp, v^\perp, w^\perp$ are pairwise distinct thin elements which together with the identity form a thin closed subset of S isomorphic to a Klein group. Conversely, if T is a triangle for which the set $1_\Omega \cup T^\perp$ is a Klein subgroup of S_1, then obviously the triangle T is exceptional. Thus we come to the following statement.

Theorem 7.2. *A triangle T is exceptional if and only if the set $1_\Omega \cup T^\perp$ is a Klein subgroup of the group S_1.* ∎

The following theorem provides the key property of non-exceptional triangles that can be used in Section 8.

Theorem 7.3. *Let $T = \{u,v,w\}$ be a non-exceptional triangle. Then there exist relations $a \in u^*w$ and $b \in w^*v$ for which $|ab \cap u^*v| = 1$.*

Proof. Pick arbitrary $a \in u^*w$ and $b \in w^*v$. Then $w \in ua \cap vb^*$. Therefore $|ua \cap vb^*| \geq 1$, and hence $|ab \cap u^*v| \geq 1$. If one of the sets u^*w or w^*v contains a thin element, then we can choose a or b to be thin. But then $|ab| = 1$ and we are done in this case. Thus for the rest of the proof we can assume that $u^*w, w^*v \subseteq S_2$. Notice that by Lemma 5.1 this implies that $w^\perp \neq u^\perp$ and $w^\perp \neq v^\perp$.

Assume now, towards a contradiction, that $|ab \cap u^*v| \geq 2$ for all $a \in u^*w$ and $b \in w^*v$. Then $2 \leq |ab \cap u^*v| \leq |u^*v| = 2$ for all a and b. So $|ab \cap u^*v| = 2$. Therefore

$$ab = u^*v, \qquad a \in u^*w,\ b \in w^*v. \tag{20}$$

If now $u^\perp = v^\perp$, then by Lemma 5.1 the set u^*v contains a thin element, say t. This implies that $t \in ab$ for every $a \in u^*v$ and a fixed $b \in w^*v$. But then u^*v consists of the unique element $a = tb^*$. Contradiction. Thus $u^\perp \neq v^\perp$, and hence $u^*v \subset S_2$. Due to (20) this shows that ab and u^*v are equal as multisets. So

$$2u^*v + \frac{2}{n_{w^\perp}} u^*w^\perp v = u^*(ww^*)v = (u^*w)(w^*v) = \sum_{a \in u^*w, b \in w^*v} ab = 4u^*v, \tag{21}$$

implying $u^*w^\perp v = n_{w^\perp} u^*v$. Suppose first that $n_{w^\perp} = 2$. Then $u^*w^\perp v = 2u^*v$. Therefore one can find points $\alpha_i, \beta_1, \beta_2 \in \Omega$ where $i = 0, \ldots, 3$ and $\beta_1 \neq \beta_2$, to have the configuration at Fig.2. However, $n_{u^*} = 2$. So either $\alpha_1 = \beta_1$ or $\alpha_1 = \beta_2$. But

$$\begin{array}{ccccccc}
 & & & \beta_1 & & & \\
 & \overset{u^*}{\nearrow} & & & \overset{v}{\searrow} & & \\
\alpha_0 & \xrightarrow{u^*} & \alpha_1 & \xrightarrow{w^\perp} & \alpha_2 & \xrightarrow{v} & \alpha_3 \\
 & \underset{u^*}{\searrow} & & & \underset{v}{\nearrow} & & \\
 & & & \beta_2 & & &
\end{array}$$

FIGURE 2.

then in any case $r(\alpha_1,\alpha_3)$ is contained in the set $vv^*\cap w^\perp$ which is empty because $v^\perp\ne w^\perp$. Contradiction. Thus $n_{w^\perp}=1, w^\perp\in S_1$ and hence by (21) we have $u^*w^\perp v=u^*v$. Therefore $\langle u^*w^\perp v,u^*v\rangle=\langle u^*v,u^*v\rangle=4$ which together with (2) implies that

$$4=\langle uu^*w^\perp,vv^*\rangle=\langle 2w^\perp+k\,u^\perp w^\perp,2\,1_\Omega+m\,v^\perp\rangle=km\langle u^\perp w^\perp,v^\perp\rangle$$

where $k,m\in\{1,2\}$. In particular, the right-hand side of the above equality is non-zero. However, since $u^\perp\ne w^\perp$ and $w^\perp\in S_1$, the latter is possible only if $u^\perp w^\perp=v^\perp$ and $k=m=2$ meaning that T is exceptional. Contradiction. ∎

8. One point extension of an algebraic isomorphism

8.1. In this section we prove the following theorem which is the key ingredient in the proof of our main result.

Theorem 8.1. *Let $\mathcal{X}$ be a non-Kleinian quasi-thin scheme of degree ≥ 9. Suppose that it has at least two orthogonals. Then any algebraic isomorphism φ from $\mathcal{X}$ to another scheme $\mathcal{X}'$ has a one point extension $\varphi_{\alpha,\alpha'}:\mathcal{X}_\alpha\to\mathcal{X}'_{\alpha'}$ for any pair of points $\alpha\in\Omega$ and $\alpha'\in\Omega'$.*

Proof. Let $\mathcal{X}=(\Omega,S)$ and let $\varphi:u\mapsto u'$ be an algebraic isomorphism from $\mathcal{X}$ to a scheme $\mathcal{X}'=(\Omega',S')$. Then it is easily seen that $\mathcal{X}'$ is quasi-thin and

$$(u^\perp)'=(u')^\perp,\quad |u^*v|=|(u')^*v'|,\qquad u,v\in S. \tag{22}$$

Let us fix points $\alpha\in\Omega$ and $\alpha'\in\Omega'$. In the following two subsections we will construct a bijection

$$\varphi':S_\alpha\to S'_{\alpha'},\ a\mapsto a', \tag{23}$$

such that $(S_\alpha(u,v))'=S'_{\alpha'}(u',v')$ for all $u,v\in S$. In Subsection 8.2 we define φ' on the union of all sets $S_\alpha(u,v)$ with $u\approx v$; in Subsection 8.3 we extend the obtained mapping on the set S_α. In Subsection 8.4 it will be proven that φ' is the (α,α')-extension of φ.

8.2. Let $u,v\in S$ be such that $S_\alpha(u,v)=S(u,v;\alpha)$. Then any $a\in S_\alpha(u,v)$ is of the form $a=w_{\alpha u,\alpha v}$ for some $w\in S$. Set

$$a'=w'_{\alpha'u',\alpha'v'}. \tag{24}$$

Then $a'\in S'(u',v';\alpha')$. Due to (22) we have $|u^*v|=|(u')^*v'|$. Therefore the mapping $a\mapsto a'$ is a bijection. It is easily seen that if $S_\alpha(u,v)=\{a,\overline{a}\}$, then $(\overline{a})'=\overline{a'}$.

Lemma 8.2. *Given a triangle $\{u,v,w\}\subset S_2$ and $a\in S_\alpha(u,v)$, $b\in S_\alpha(v,w)$, $c\in S_\alpha(u,w)$ we have*

$$a\cdot b=c\ \Rightarrow\ a'\cdot b'=c'.$$

Proof. Assume first that the triangle $\{u,v,w\}$ is non-exceptional. Then by Theorem 7.3 there exist relations $x\in u^*w$, $y\in w^*v$ and $z\in u^*v$ such that $xy\cap u^*v=\{z\}$. This implies that $x'y'\cap u'^*v'=\{z'\}$, and $a_1\cdot b_1=c_1$ and $a'_1\cdot b'_1=c'_1$ where

$$a_1=x_{\alpha u,\alpha w},\quad b_1=y_{\alpha w,\alpha v},\quad c_1=z_{\alpha u,\alpha v}$$

Now let $a\in S_\alpha(u,v)$, $b\in S_\alpha(v,w)$, $c\in S_\alpha(u,w)$ be such that $a\cdot b=c$. There the pair (a,b) is one of the following: (a_1,b_1), $(\overline{a}_1,b_1)$, $(a_1,\overline{b}_1)$ or $(\overline{a}_1,\overline{b}_1)$. In the

first case the statement is clear. In the second one we have $a' = (\overline{a}_1)' = \overline{a_1'}$. By Lemma 7.1 this implies that $c = a \cdot b = \overline{c}_1$. Thus and

$$a' \cdot b' = \overline{a_1'} \cdot b_1' = \overline{a_1' \cdot b_1'} = \overline{c_1'} = \overline{c_1}' = c'.$$

The remaining two cases are considered in a similar manner.

Let now $T = \{u, v, w\}$ be an exceptional triangle. Since $\mathcal{X}$ is a non-Kleinian scheme, this implies that $|S^\perp| \ge 4$. Therefore there exists a thick basic relation t such that $t^\perp \notin T^\perp$. Then each set $\{t, x, y\}$ where x and y are distinct elements of T, is a non-exceptional triangle (otherwise $t^\perp = x^\perp \cdot y^\perp \in T^\perp$). Therefore there exist $x \in S_\alpha(t, u)$, $y \in S_\alpha(t, v)$ and $z \in S_\alpha(t, w)$ for which $x^* \cdot y = a$, $y^* \cdot z = b$ and $x^* \cdot z = c$. Since the corresponding triangles are non-exceptional, from the first part of the proof it follows that

$$x'^* \cdot y' = a', \quad y'^* \cdot z' = b', \quad x'^* \cdot z' = c'.$$

Therefore $a'b' = (x'^* \cdot y') \cdot (y'^* \cdot z') = x'^* \cdot z' = c'$. The lemma is proved. ■

8.3. Let $u, v \in S$ be such that $S_\alpha(u, v) \neq S(u, v; \alpha)$. Then by Lemma 6.3 the relations u and v are non-adjacent and $u^\perp = v^\perp$. Since the scheme $\mathcal{X}$ has at least two orthogonals, one can find a relation $w = w_{u,v}$ for which $w^\perp \neq u^\perp$. Then by Lemma 6.3 we have

$$S_\alpha(u, v) = S(u, w; \alpha) \cdot S(w, v; \alpha). \tag{25}$$

Here $u \approx w$ and $w \approx v$, and $S_\alpha(u, w) = S(u, w; \alpha)$ and $S_\alpha(w, v) = S(w, v; \alpha)$. Therefore one can consider two bijections

$$S_\alpha(u, w) \to S'_{\alpha'}(u', w'),\ b \mapsto b', \qquad S_\alpha(w, v) \to S'_{\alpha'}(w', v'),\ c \mapsto c'$$

that were defined in Subsection 8.2. Now for a fixed $b \in S_\alpha(u, w)$ and for any $a \in S_\alpha(u, v)$ there exists a uniquely determined $c \in S_\alpha(w, v)$ such that $a = b \cdot c$. Set

$$a' = b' \cdot c'. \tag{26}$$

Then $a' \in S'(u', w'; \alpha') \cdot S'(w', v'; \alpha') = S'_{\alpha'}(u', v')$ and the mapping $a \mapsto a'$ is a required bijection.

Lemma 8.3. *In the above notation set $w_1 = w$, $b_1 = b$ and $c_1 = c$. Then given $w_2 \in S_2$ with $w_2^\perp \neq u^\perp$, $b_2 \in S_\alpha(u, w_2)$ and $c_2 \in S_\alpha(w_2, u)$ we have*

$$b_1 \cdot c_1 = b_2 \cdot c_2 \ \Rightarrow\ b_1' \cdot c_1' = b_2' \cdot c_2'.$$

Proof. Without loss of generality we can assume that $w_1 \neq w_2$. In the above assumptions the element $u^\perp = v^\perp$ is not equal neither to $w_1^\perp$ nor to $w_2^\perp$. Therefore $u \approx w_1$, $w_1 \approx v$, $v \approx w_2$ and $w_2 \approx u$.

Suppose that $w_1 \approx w_2$. It follows from the equality $b_1 \cdot c_1 = b_2 \cdot c_2$ that the relation $d := b_2^* \cdot b_1 = c_2 \cdot c_1^*$ belongs to the set $S_\alpha(w_2, w_1)$. Then $b_1 = b_2 \cdot d$ and $c_1 = d^* \cdot c_2$. Since both $\{w_1, w_2, u\}$ and $\{w_1, w_2, v\}$ are triangles, Lemma 8.2 implies

$$b_1' \cdot c_1' = (b_2 \cdot d)' \cdot (d^* \cdot c_2)' = b_2' \cdot d' \cdot (d')^* \cdot c_2' = b_2' \cdot c_2'$$

and we are done.

Let now $w_1 \not\approx w_2$. Then $w_1^\perp = w_2^\perp$ and this element is thin (Lemma 5.1). By Lemma 5.5 applied to $T = \{u, v, w_1, w_2\}$ there exists a relation $t \in S_2$ such that

any set $\{t, x, y\}$ with $x \in \{u, v\}$ and $y \in \{w_1, w_2\}$ is a triangle. Pick an arbitrary $a_1 \in S_\alpha(u, t)$. Then we have

$$d_1 := a_1^* \cdot b_1 \in S_\alpha(t, w_1), \qquad d_2 := a_1^* \cdot b_2 \in S_\alpha(t, w_2), \qquad a_2 := d_1 \cdot c_1 \in S_\alpha(t, v).$$

Therefore $a_1 \cdot a_2 = (a_1 \cdot d_1) \cdot (d_1^* \cdot a_2) = b_1 \cdot c_1 = b_2 \cdot c_2 = a_1 \cdot d_2 \cdot c_2$. This implies that $c_2 = d_2^* \cdot a_2$ (Figure 3). Thus by Lemma 8.2 we conclude that

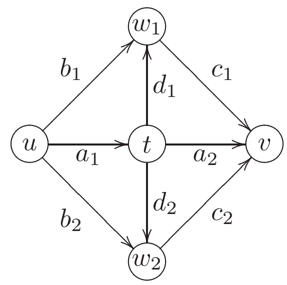

Figure 3.

$$b_1' \cdot c_1' = (a_1 \cdot d_1)' \cdot (d_1^* \cdot a_2)' = a_1' \cdot (d_1' \cdot (d_1')^*) \cdot a_2' = a_1' \cdot a_2' =$$
$$(b_2 \cdot d_2^*)' \cdot (d_2 \cdot c_2)' = b_2' \cdot (d_2' \cdot (d_2^*)') \cdot c_2' = b_2' \cdot c_2'$$

which completes the proof. ∎

8.4. For the bijection φ' defined in Subsections 8.2 and 8.3 the conditions (5) are obviously satisfied. Thus to check that φ' is the (α, α')-extension of φ it suffices to verify only that

$$(b \cdot c)' = b' \cdot c', \qquad b, c \in S_\alpha,\ b \cdot c \neq \emptyset. \tag{27}$$

Let $b, c \in S_\alpha$ be such that $b \cdot c \neq \emptyset$. Then there exist $u, v, w \in S$ such that $b \subset \alpha u \times \alpha v$ and $c \subset \alpha v \times \alpha w$. Without loss of generality we can assume that $u, v, w \in S_2$. If the cardinality of the set $T = \{u, v, w\}$ is less than three, then at least one of the relations $b, c, b \cdot c$ is an in-fiber relation and we are done by Lemma 7.1. So we may assume that $|T| = 3$.

First we notice that (27) is correct when $u \approx v$ and $v \approx w$. Indeed, if in addition u and w are adjacent, then T is a triangle and we are done by Lemma 8.2; otherwise u and w are non-adjacent and we are done by Lemma 8.3. Thus we can assume that at least one of the pairs $\{u, v\}$, $\{v, w\}$ is non-adjacent. By Lemma 5.1 this implies that

$$|T^\perp| \leq 2.$$

Now Lemma 5.5 applied to $T = \{u, v, w\}$ implies that there exists a relation $t \in S_2 \setminus T$ such that any set $\{t, x, y\}$ with $x, y \in T$, is a triangle. As we have shown before, the condition (27) holds for any $b \in S_\alpha(x, t)$ and $c \in S_\alpha(t, y)$. On the other hand, by Corollary 6.4 the coherent configuration S_α is 1-regular. Thus there exist relations $a_1 \in S_\alpha(u, t)$, $a_2 \in S_\alpha(t, v)$ and $a_3 \in S_\alpha(t, w)$ such that $b = a_1 \cdot a_2$ and $c = a_2^* \cdot a_3$ (see Fig. 4). Since $|u^* t| = |t^* v| = 2$, the condition (27) holds for a_1 and a_2 implying $b' = a_1' \cdot a_2'$. Analogously, $c' = (a_2^*)' \cdot a_3'$ and $(a_1 \cdot a_3)' = a_1' \cdot a_3'$. Therefore

$$(b \cdot c)' = (a_1 \cdot a_2 \cdot a_2^* \cdot a_3)' = (a_1 \cdot a_3)' = a_1' \cdot a_3' =$$

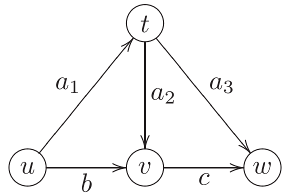

FIGURE 4.

$$a_1' \cdot a_2' \cdot (a_2^*)' \cdot a_3' = (a_1 \cdot a_2)' \cdot (a_2^* \cdot a_3)' = b' \cdot c'$$

which completes the proof of (27). Theorem 8.1 is proven. ∎

9. Proofs of the main results

In this section $\mathcal{X} = (\Omega, S)$ denotes a quasi-thin scheme and we write 1 instead of 1_Ω.

9.1. **Proof of Theorem 1.1.** Suppose first that $\mathcal{X}$ is non-Kleinian. We have to prove that it is schurian and separable. By Corollary 5.3 and Theorem 5.2 we can assume that $\mathcal{X}$ is of degree ≥ 9 and has at least two orthogonals. Then by Theorem 8.1 any algebraic isomorphism from $\mathcal{X}$ to another scheme has one point extension at every pair of points. Thus the scheme $\mathcal{X}$ is schurian and separable by Theorem 6.5.

Suppose that the scheme $\mathcal{X}$ is Kleinian. Denote by T its thin residue. Then the thin residue extension $\mathcal{X}_0 = \mathcal{X}_{(T)}$ is a Klein configuration. For this configuration we keep the notation of Section 4.1 with $G = T$ and the group isomorphisms (7) taking $g \in T$ to $g_i = g_{\Omega_i, \Omega_i}$, $i \in I$; in particular, $|I| = |\Omega/T|$. Let

$$\Psi = \{\psi_f : \ f \in \mathrm{Aut}(\mathcal{X}_{\Omega/T})\}$$

be the group of algebraic automorphisms ψ_f of the scheme $\mathcal{X}$ defined in statement (i) in the proof of Theorem 3.3. Then Ψ acts regularly on the set $\mathrm{Fib}(\mathcal{X}_0)$, and hence on the set I so that $i^\psi = j$ if and only if $(\Omega_i)^\psi = \Omega_j$. By the choice of isomorphisms (7) for any $i, j \in I$ and $s \in S$ we have

$$(s_{ij} \cdot g_j)^\psi = s_{ij}^\psi \cdot (g_j)^\psi = s_{i^\psi j^\psi} \cdot g_{j^\psi}, \qquad g \in G, \ \psi \in \Psi,$$

where $s_{ij} = s_{\Omega_i, \Omega_j}$. This implies that $R_{ij} = R_{i^\psi, j^\psi}$ for all i, j where R_{ij} is the group defined in (8) for $\mathcal{X} = \mathcal{X}_0$. Thus

$$R(\mathcal{X}_0, G)^\Psi = R(\mathcal{X}_0, G). \tag{28}$$

We note that no entry of this matrix equal to G. Indeed, otherwise from the condition (K1) it follows that $\mathcal{X}_0$ contains a basic relation $s = \Omega_i \times \Omega_j$ for some $i, j \in I$. However, then the basic relation of the scheme $\mathcal{X}$ that contains s has valency ≥ 4 which is impossible because the scheme $\mathcal{X}$ is quasi-thin. Thus the hypothesis of Corollary 4.4 is satisfied.

Lemma 9.1. *The isomorphism type of the linear space $\mathcal{G}_J = \mathcal{G}((\mathcal{X}_0)_J)$ does not depend on the transversal $J \in \mathcal{J}(\mathcal{X}_0, G)$. Moreover, $\mathcal{G}_J$ is isomorphic to either near-pencil on 3 points or a projective or affine plane of order 2.*

Proof. Let J be a transversal of the partition of the set I in the classes of the equivalence relation $\sim$. Then the linear space $\mathcal{G}_J$ has at least two lines. Indeed, suppose on the contrary that $L_i(H) = I$ for some element $i \in I$ and a group $H \le G$ of order 2. Then

$$R_{ji} = H, \qquad j \in I \setminus \{i\},$$

where $R = R(\mathcal{X}_0, G)$. Due to (28) this implies that any non-diagonal entry of the matrix R equals to H. Therefore the scheme $\mathcal{X}$ has a unique orthogonal g_i^Ψ where g is the element of H of order 2. However, this is impossible because $|S^\perp| \ge 2$. Thus by Corollary 4.4 the linear space $\mathcal{G}_J$ is a projective or affine plane of order 2, or $\mathcal{G}_J$ is one of the four linear spaces at Fig.1.

Given nonnegative integers d, e denote by $M_{d,e}$ the set of all pairs $(i, J) \in I \times \mathcal{J}(\mathcal{X}_0, G)$ such that $i \in J$ and the linear space $\mathcal{G}_J$ contains exactly d (resp. e) lines of size 2 (resp. of size 3) that are incident to i. Suppose that $(i, J) \in M_{d,e}$. Then from condition (10) it follows that $\{i\} \times \mathcal{J}_i(\mathcal{X}_0, G) \subset M_{d,e}$ where $\mathcal{J}_i(\mathcal{X}_0, G)$ is the set of all transversals $J \in \mathcal{J}(\mathcal{X}_0, G)$ containing i. Due to (28) this implies that

$$\bigcup_{i \in I} \{i\} \times \mathcal{J}_i(\mathcal{X}_0, G) \subset M_{d,e}.$$

Thus for any $J \in \mathcal{J}(\mathcal{X}_0, G)$ the linear space $\mathcal{G}_J$ contains exactly d (resp. e) lines of size 2 (resp. of size 3) through any point. However, this is possible only if $d = e$ and this number is 2 or 3. In the former case $\mathcal{G}_J$ is the first linear space at Fig.1 or an affine plane of order 2, whereas in the latter case $\mathcal{G}_J$ is a projective plane of order 2. Since all these geometries have distinct number of points, we are done. ∎

Depending on the isomorphism type of linear spaces $\mathcal{G}_J$ we will say that the scheme $\mathcal{X}$ is a scheme over near-pencil, affine plane or projective plane. It should be noted that the number of points in $\mathcal{G}_J$ coincides with the number $|J| = |\Omega|/|S_1|$ which was called the index of $\mathcal{X}$ in the introduction.

Theorem 9.2. *Any quasi-thin Klein scheme $\mathcal{X}$ is of index $3, 4$ or 7; in these cases $\mathcal{X}$ is a scheme over near-pencil, affine plane or projective plane. Moreover, in the former case $\mathcal{X}$ is schurian and separable, whereas in the latter case $\mathcal{X}$ is not commutative.*

Proof. The first statement immediately follows from Lemma 9.1. A straightforward computation shows that any Klein configuration on 12 points is schurian and separable. Therefore the schurity and separability of a Kleinian quasi-thin scheme over near-pencil follows from Theorem 3.3 and Lemma 4.2. To prove the last statement we observe that the commutativity of the scheme $\mathcal{X}$ implies that the matrix $R(\mathcal{X}_0, G)$ is symmetric. Therefore either a linear space $\mathcal{G}_J$ contains exactly one line or it has two disjoint lines. Since both of these possibilities are impossible when $\mathcal{G}_J$ is a projective plane, we are done. ∎

By Theorem 9.2 to complete the proof we have to verify that given $i \in \{4, 7\}$ there exist infinitely many both non-schurian and non-separable Kleinian schemes of index i. To do this we denote by $\mathcal{X}_{16}$ and $\mathcal{X}'_{16}$ (resp. $\mathcal{X}_{28}$ and $\mathcal{X}'_{28}$) the schemes #173 and #172 (resp. #176 and #175) from the Hanaki-Miyamoto list [9] of association schemes of degree 16 (resp. of degree 28). A straightforward computation shows that:

(1) all the schemes $\mathcal{X}_{16}$, $\mathcal{X}'_{16}$, $\mathcal{X}_{28}$ and $\mathcal{X}'_{28}$ are quasi-thin and Kleinian; the former two are of index 4 whereas the latter two are of index 7,

(2) the schemes $\mathcal{X}'_{16}$ and $\mathcal{X}'_{28}$ are schurian whereas the schemes $\mathcal{X}_{16}$ and $\mathcal{X}_{28}$ are non-schurian,
(3) the scheme $\mathcal{X}_{16}$ is algebraically isomorphic to the scheme $\mathcal{X}'_{16}$, the scheme $\mathcal{X}_{28}$ is algebraically isomorphic to the schemed $\mathcal{X}'_{28}$.

Thus $\mathcal{X}_{16}$ and $\mathcal{X}_{32}$ are non-schurian and non-separable quasi-thin Klein schemes of indices 4 and 7. Let $\mathcal{X}$ be one of this scheme and let $\mathcal{Y}$ be an arbitrary regular scheme. Then obviously $\mathcal{X}\otimes\mathcal{Y}$ is a quasi-thin Klein scheme of the same index as $\mathcal{X}$. Since by Theorem 3.2 the scheme $\mathcal{X}\otimes\mathcal{Y}$ is also non-schurian and non-separable, we are done. ∎

9.2. **Proof of Theorem 1.3.** Suppose that the scheme $\mathcal{X}$ is commutative. To prove that it is schurian by Theorem 1.1 we can assume that this scheme is Kleinian. Then from Lemma 5.4 it follows that

$$S^{\perp}=\{a,b,c\}$$

where $a,b,c\in S_1$ with $a^2=b^2=c^2=1$ and $ab=c$, and hence

$$S=S_1\cup S_a\cup S_b\cup S_c. \tag{29}$$

Moreover, given $e\in\{a,b,c\}$ we have $S_e=S_1S_e=S_eS_1$. Choose elements $x\in S_a$, $y\in S_b$, $z\in S_c$ so that $1\in xyz$. Then

$$\begin{array}{rcl} xy &=& z^*+z^*a=z^*+z^*b,\\ yz &=& x^*+x^*b=x^*+x^*c,\\ zx &=& y^*+y^*c=y^*+y^*a.\end{array} \tag{30}$$

It follows from $x^*\in S_a, y^*\in S_b, z^*\in S_c$ that there exist $u,v,w\in S_1$ such that

$$x^*=xu^*,\qquad y^*=yv^*,\qquad z^*=zw^*. \tag{31}$$

Since $x=xa, y=yb, z=zc$, there is a certain freedom in a choice of u,v,w. More precisely, we can always replace (if necessary) u by ua, v by vb and w by wc. All these replacements could be done independently.

Applying $*$ to the first row of (30) we obtain that $x^*y^*=z+za$. By (31) this implies that

$$z^*+z^*a=xy=z^*(uvw)+z^*a(uvw).$$

Therefore $uvw\in\{1,a,b,c\}$. If $uvw=a$, then by replacing u by ua we obtain that $uvw=1$ (the same could be done in the cases $uvw=b$ or $uvw=c$). Thus in what follows we can assume that $uvw=1$.

Let $\alpha,\beta,\gamma\in\Omega$ be such that $(\alpha,\beta)\in x$, $(\beta,\gamma)\in y$ and $(\gamma,\alpha)\in z$. Since $c_{xy}^{z^*}=c_{yx}^{z^*}=1$, there exists a unique point $\delta\in\Omega$ such that $(\alpha,\delta)\in x$ and $(\delta,\gamma)\in y$. The pair (δ,β) belongs to the relation

$$yx^*=yxu^*=z^*u^*+z^*u^*b=zw^*u^*+zw^*u^*b=zv+zvb,$$

and hence belongs to either zv or zvb. The latter case may be reduced to the first one by the replacement $u\leftrightarrow ua$, $v\leftrightarrow vb$, $w\leftrightarrow wc$. Notice that this replacement keeps invariant the relations $1\in xyz$ and $uvw=1$. Thus we may assume that $(\delta,\beta)\in zv$. This yields us the picture on Fig. 5. Thus $r(\lambda,\mu)\in S_2$ for all distinct elements λ,μ in the set $\Lambda=\{\alpha,\beta,\gamma,\delta\}$. Therefore due to (29) the set Ω is a disjoint union of the sets λS_1 where λ runs over Λ. This enables us to identify the sets Ω

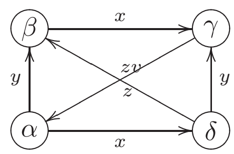

FIGURE 5.

and $S_1 \times \Lambda$ so that the adjacency matrices $X, Y, Z \in \mathrm{Mat}_\Lambda(\mathbb{Z}S_1)$ of the relations x, y, z take the following forms:

$$X = \begin{pmatrix} 0 & 0 & 0 & A \\ 0 & 0 & A & 0 \\ 0 & uA & 0 & 0 \\ uA & 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & B & 0 & 0 \\ vB & 0 & 0 & 0 \\ 0 & 0 & 0 & vB \\ 0 & 0 & B & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & wC & 0 \\ 0 & 0 & 0 & wvC \\ C & 0 & 0 & 0 \\ 0 & v^*C & 0 & 0 \end{pmatrix}$$

where $A = 1 + a$, $B = 1 + b$ and $C = 1 + c$. Let us define the permutations f, g and h of the set $\Omega = S_1 \times \Lambda$ by means of their permutation matrices:

$$F = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & u & 0 & 0 \\ u & 0 & 0 & 0 \end{pmatrix}, G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ v & 0 & 0 & 0 \\ 0 & 0 & 0 & v \\ 0 & 0 & 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 0 & 0 & w & 0 \\ 0 & 0 & 0 & wv \\ 1 & 0 & 0 & 0 \\ 0 & v^* & 0 & 0 \end{pmatrix}.$$

Using the equality $uvw = 1$ we obtain that $FGH = I$ (the identity matrix). Also

$$X = FA, \quad Y = GB, \quad Z = HC.$$

A direct calculation shows that F an G commute. Therefore $H = F^*G^*$ commute with F and G. This implies that F, G, H commute with X, Y, Z. Therefore $f, g, h \in \mathrm{Aut}(\Omega, S)$. Moreover it follows from $F^2 = uI$, $G^2 = vI$, $H^2 = wI$ that $\langle S_1, f, g, h \rangle$ is a regular abelian group of automorphisms of S. So the group $\mathrm{Aut}(\Omega, S)$ is transitive and we are done by Theorem 6.2.

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Netanya Academic College, Netanya, Israel
E-mail address: `muzy@netanya.ac.il`

Steklov Institute of Mathematics at St. Petersburg, Russia
E-mail address: `inp@pdmi.ras.ru`